

EXTENSION OF WIENER-WINTNER DOUBLE RECURRENCE THEOREM TO POLYNOMIALS

IDRIS ASSANI AND RYO MOORE

ABSTRACT. We extend our result on the convergence of double recurrence Wiener-Wintner averages to the case where we have a polynomial exponent. We will show that there exists a single set of full measure for which the averages

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) \phi(p(n))$$

converge for any polynomial p with real coefficients, and any continuous function $\phi : \mathbb{T} \rightarrow \mathbb{C}$. We also show that if either function belongs to an orthogonal complement of an appropriate Host-Kra-Ziegler factor that depends on the degree of the polynomial p , then the averages converge to zero uniformly for all polynomials. This paper combines the authors' previously announced work.

NOTATIONS/CONVENTIONS

Unless specified otherwise, the following notations/conventions will be used throughout the paper.

- $\mathbb{R}_k[\xi]$ is a collection of degree- k polynomials with real coefficients, whereas $\mathbb{R}[\xi]$ is the collection of all the polynomials with real coefficients.
- \mathcal{Z}_k is the k -th Host-Kra-Ziegler factor (cf. [15, 21]), whereas $||| \cdot |||_{k+1}$ is the Gowers-Host-Kra seminorm (cf. [14, 15]) that characterizes the factor \mathcal{Z}_k .
- Sometimes, we denote $e(\alpha) = e^{2\pi i \alpha}$.
- Functions f_1 and f_2 are taken to be real-valued. The results presented here can be easily extended to the case where they are complex-valued functions.
- When we have $A \lesssim_B C$, we mean that $A \leq DC$ for some real constant $D > 0$ that depends on B .

1. INTRODUCTION

The following extension on Bourgain's pointwise result on double recurrence [10] was proven in [4].

Theorem 1.1. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system, $a, b \in \mathbb{Z}$ such that $a \neq b$, and $f_1, f_2 \in L^\infty(\mu)$. Let*

$$W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}.$$

- (1) *(Double Uniform Wiener-Wintner Theorem) If either f_1 or f_2 belongs to \mathcal{Z}_2^\perp , then there exists a set of full measure X_{f_1, f_2} such that for all $x \in X_{f_1, f_2}$,*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, t)| = 0.$$

- (2) *(General Convergence) If $f_1, f_2 \in \mathcal{Z}_2$, then for μ -a.e. $x \in X$, $W_N(f_1, f_2, x, t)$ converges for all $t \in \mathbb{R}$.*

One of the estimates that was established to prove the uniform convergence result above was the following (this was obtained in the proof of [4, Theorem 5.1]):

Theorem 1.2. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system, and $f_1, f_2 \in L^\infty(\mu)$. We have*

$$(1) \quad \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t} \right|^2 d\mu(x) \lesssim_{a,b} \min \left\{ \|f_1\|_3^2, \|f_2\|_3^2 \right\}.$$

In this paper, we will extend Theorem 1.1 to the case where we have a polynomial exponent, i.e. we will show that there exists a set of full measure X_{f_1, f_2} such that for any $x \in X_{f_1, f_2}$, the averages

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n))$$

converge for all polynomials p with real coefficients. Furthermore, we will show that an appropriate Host-Kra-Ziegler factor, depending on the degree of the polynomial p , is a characteristic factor for these averages. The statement is given precisely as follows:

Theorem 1.3. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system, $a, b \in \mathbb{Z}$ such that $a \neq b$, and $f_1, f_2 \in L^\infty(\mu)$. Let*

$$(2) \quad W_N(f_1, f_2, x, p) = \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)),$$

where p is a polynomial with real coefficients. Then the following are true:

- (1) *If either f_1 or f_2 belongs to \mathcal{Z}_{k+1}^\perp , then there exists a set of full measure X_{f_1, f_2} such that for all $x \in X_{f_1, f_2}$,*

$$\limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_k[\xi]} |W_N(f_1, f_2, x, p)| = 0.$$

- (2) *If $f_1, f_2 \in \mathcal{Z}_{k+1}$, then for μ -a.e. $x \in X$, the averages $W_N(f_1, f_2, x, p)$ converge for all $p \in \mathbb{R}_k[\xi]$.*

- (3) *There exists a set of full measure X_{f_1, f_2} such that for all $x \in X_{f_1, f_2}$, the averages*

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) \phi(p(n))$$

converge for all continuous functions $\phi : \mathbb{T} \rightarrow \mathbb{C}$ and polynomials p with real coefficients.

We remark that when $a = b$, the averages in (2) become

$$W_N(f_1, f_2, x, p) = \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_2)(T^{an}x) e^{2\pi i p(n)}.$$

which reduces to the polynomial extension of the pointwise ergodic theorem, and some results on a convergence of this type of averages have been achieved by E. Lesigne [19, 20] and N. Frantzikinakis [13]. Lesigne showed that for any ergodic system (X, \mathcal{F}, μ, T) , there exists a set of full measure for which the

averages

$$(3) \quad \frac{1}{N} \sum_{n=1}^N \phi(p(n)) f(T^n x)$$

converge for all polynomials p and a continuous function $\phi : \mathbb{T} \rightarrow \mathbb{C}$. Furthermore, if T is assumed to be totally ergodic, p is a k -th degree polynomial, and f belongs to the orthogonal complement of the k -th degree Abramov factor, then the averages in (3) converge to 0. Frantzikinakis extended this result by showing the uniform counterpart: Assuming that T is totally ergodic and f belongs to the orthogonal complement of the k -th degree Abramov factor, then

$$\lim_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_k[\xi]} \left| \frac{1}{N} \sum_{n=1}^N \phi(p(n)) f(T^n x) \right| = 0.$$

Frantzikinakis also showed that the assumption T being totally ergodic cannot be replaced with T merely ergodic by providing a counterexample.

The classical Wiener-Wintner averages are also generalized to the cases where $e(nt)$ is replaced by a nilsequence (cf. [9, 16]). For instance, B. Host and B. Kra showed [16, Theorem 2.22] that given an ergodic system (X, \mathcal{F}, μ, T) and a function $f \in L^\infty(\mu)$, there exists a set of full measure $X_f \subset X$ such that for any $x \in X_f$, and for any nilsequence (b_n) , the averages

$$\frac{1}{N} \sum_{n=1}^N f(T^n x) b_n$$

converge. Furthermore, the uniform version of this result was obtained by T. Eisner and P. Zorin-Kranich [12, Theorem 1.2]. We note that some aspects of polynomial Wiener-Wintner averages are covered in their studies because if p is any degree- k polynomial, then $b_n = e(p(n))$ is a k -step nilsequence.

Recently, T. Eisner and B. Krause obtained a uniform Wiener-Wintner results for averages with weights involving Hardy functions and for "twisted" polynomial ergodic averages [11].

1.1. Remarks. This paper combines the authors' previous preprints that appeared during the summer of 2014 [5, 6]. In this paper, we will focus on the stronger result obtained in [6] (which is Theorem 1.3 of the current paper), and use that to prove the result obtained in [5] (which is Corollary 3.1 of the current paper). It is worth mentioning that Corollary 3.1 was obtained before Theorem 1.3 using different machineries.

Since the first submission of this paper in September 2014, some new results have been announced. For instance, the authors showed that the sequence $a_n = f_1(T^{an}x) f_2(T^{bn}x)$ is a good universal weight for the Furstenberg averages in the L^2 -norm for μ -a.e. $x \in X$ [7], which was extended further to the case where we have commuting transformations [8]. The first author extended the double recurrence Wiener-Wintner result to nilsequences [3], and a result similar to this was announced by P. Zorin-Kranich independently [22].

2. PROOF OF THEOREM 1.3

To prove Theorem 1.3, we break the proof in two cases: The case where either f_1 or f_2 belongs to \mathcal{Z}_{k+1}^\perp , and the case where both f_1 and f_2 belong to \mathcal{Z}_{k+1} , where k is the degree of the polynomial p .

The first case corresponds exactly to (1) of Theorem 1.3, and we prove this by applying induction on k . We notice that the base case $k = 1$ is essentially (1) of Theorem 1.1, where $p(n) = tn$. For the inductive step, we first apply van der Corput's lemma to reduce the degree of the polynomial, which allows us to use the inductive hypothesis. Then we use the estimate similar to Theorem 1.2 for the polynomials of higher degree to control the integral of the limit supremum of these averages. Consequently, the uniform Wiener-Wintner result follows.

For the second case, we restrict ourselves to the case where both functions are measurable with respect to \mathcal{Z}_{k+1} . Using the fact that the $k + 1$ -th Host-Kra-Ziegler factor is an inverse limit of a sequence of $k + 1$ -step nilsystems that are factors of (X, \mathcal{F}, μ, T) (cf. [15, Theorem 10.1]), we further restrict ourselves to the case where (X, \mathcal{F}, μ, T) is an ergodic nilsystem. Furthermore, since the set of continuous functions is dense in $L^2(\mu)$, we will assume that f_1 and f_2 are continuous. These assumptions allow us to use Leibman's pointwise convergence result [18] to show that the averages converge for all $x \in X$.

Combining these two cases, we prove (3) of Theorem 1.3.

2.1. Proof of (1) of Theorem 1.3. In this section, we prove (1) of Theorem 1.3. One of the main inequalities used in this part of the proof is van der Corput's lemma, which is stated as follows (a proof can be found in [17]):

Lemma 2.1 (van der Corput). *If (a_n) is a sequence of complex numbers and if H is an integer between 0 and $N - 1$, then*

$$(4) \quad \left| \frac{1}{N} \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{N+H}{N^2(H+1)} \sum_{n=0}^{N-1} |a_n|^2 + \frac{2(N+H)}{N^2(H+1)^2} \sum_{h=1}^H (H+1-h) \operatorname{Re} \left(\sum_{n=0}^{N-h-1} a_n \bar{a}_{n+h} \right).$$

The next lemma addresses the measurability of the map that takes a point in the phase space to the supremum of the polynomial Wiener-Wintner averages over a collection of polynomials with the same degree.

Lemma 2.2. *For each positive integers N and k , the map*

$$x \in X \mapsto F_{N,k}(x) = \sup_{p \in \mathbb{R}_k[\xi]} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|$$

is measurable.

Proof. If we denote $p(n) = \sum_{j=0}^k c_j n^j$, then

$$\begin{aligned} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| &= \sup_{(c_0, c_1, c_2, \dots, c_k) \in \mathbb{R}^{k+1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e\left(\sum_{j=0}^k c_j n^j\right) \right| \\ &= \sup_{(c_0, c_1, c_2, \dots, c_k) \in \mathbb{Q}^{k+1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e\left(\sum_{j=0}^k c_j n^j\right) \right|, \end{aligned}$$

where the last equality follows from the fact that \mathbb{Q}^{k+1} is dense in \mathbb{R}^{k+1} , and the map

$$(c_0, c_1, c_2, \dots, c_k) \mapsto e\left(\sum_{j=0}^k c_j n^j\right)$$

is a continuous one from \mathbb{R}^{k+1} to \mathbb{T} for each $n \in \mathbb{Z}$. Since \mathbb{Q}^{k+1} is countable, it follows that the map $x \mapsto F_{N,k}(x)$ is measurable for each k and N . \square

2.1.1. *Proof for the case $k = 2$.* To better illustrate the proof of Theorem 1.3(1), we first prove this result for the case where $p(n) = \alpha n^2 + \beta n$. Suppose that either f_1 or f_2 belongs to the orthogonal complement of \mathcal{Z}_3 . Then we apply van der Corput's lemma and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} &\limsup_{N \rightarrow \infty} \sup_{\alpha, \beta \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(\alpha n^2 + \beta n) \right|^2 \\ &\leq \frac{2}{H} + \frac{4}{(H+1)^2} \sum_{h=1}^H (H+1-h) \\ &\quad \cdot \limsup_{N \rightarrow \infty} \sup_{\alpha, \beta \in \mathbb{R}} \operatorname{Re} \left(\frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(-(\alpha h^2 + 2\alpha hn + \beta h)) \right) \\ &\leq \frac{2}{H} + 4 \left(\frac{1}{H+1} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \sup_{\alpha \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(-2\alpha hn) \right|^2 \right)^{1/2}. \end{aligned}$$

We integrate both sides of the inequality above (which can be done by Lemma 2.2), and by Hölder's inequality, we obtain

(5)

$$\begin{aligned} &\int \limsup_{N \rightarrow \infty} \sup_{\alpha, \beta \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(\alpha n^2 + \beta n) \right|^2 d\mu(x) \\ &\leq \frac{2}{H} + 4 \left(\frac{1}{H+1} \sum_{h=1}^H \int \limsup_{N \rightarrow \infty} \sup_{\alpha \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(-2\alpha hn) \right|^2 d\mu(x) \right)^{1/2}. \end{aligned}$$

Note that the inside of the integral on the right hand side of (5) is a double recurrence Wiener-Wintner average (by setting $t = -2\alpha h$) for each h . By (1), we have

$$\int \limsup_{N \rightarrow \infty} \sup_{\alpha \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(-2\alpha hn) \right|^2 d\mu(x)$$

$$\lesssim_{a,b} \min \left\{ \left\| \|f_1 \cdot f_1 \circ T^{ah}\|_3^2, \left\| \|f_2 \cdot f_2 \circ T^{bh}\|_3^2 \right\|_3 \right\}.$$

Therefore,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{\alpha, \beta \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(\alpha n^2 + \beta n) \right|^2 d\mu(x) \\ & \lesssim_{a,b} \frac{1}{H} + \min \left\{ \left(\frac{1}{H} \sum_{h=1}^H \left\| \|f_1 \cdot f_1 \circ T^{ah}\|_3^2 \right\|_3 \right)^{1/2}, \left(\frac{1}{H} \sum_{h=1}^H \left\| \|f_2 \cdot f_2 \circ T^{bh}\|_3^2 \right\|_3 \right)^{1/2} \right\} \\ & \lesssim_{a,b} \frac{1}{H} + \min \left\{ \left(\frac{1}{H} \sum_{h=1}^H \left\| \|f_1 \cdot f_1 \circ T^{ah}\|_3^4 \right\|_3 \right)^{1/4}, \left(\frac{1}{H} \sum_{h=1}^H \left\| \|f_2 \cdot f_2 \circ T^{bh}\|_3^4 \right\|_3 \right)^{1/4} \right\}, \end{aligned}$$

and by letting $H \rightarrow \infty$, we obtain

$$(6) \quad \int \limsup_{N \rightarrow \infty} \sup_{\alpha, \beta \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(\alpha n^2 + \beta n) \right|^2 d\mu(x) \lesssim_{a,b} \min \left\{ \|f_1\|_4^2, \|f_2\|_4^2 \right\}.$$

Since either f_1 or f_2 belongs to \mathcal{Z}_3^\perp , either $\|f_1\|_4$ or $\|f_2\|_4$ equals 0. This completes the proof for the case where $p(n) = \alpha n^2 + \beta n$.

2.1.2. The proof for any positive integer k . One of the key inequalities (besides van der Corput's lemma) used for the case where $k = 2$ is (6), where we controlled the integral of the limsup of the averages by an appropriate Gowers-Host-Kra seminorm. We generalize this inequality for polynomials with higher degree with induction (on k) and van der Corput's inequality.

Lemma 2.3. *Let (X, \mathcal{F}, μ, T) be an ergodic system, and $f_1, f_2 \in L^\infty$. Then*

$$(7) \quad \int \limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|^2 d\mu(x) \lesssim_{a,b,k} \min \left\{ \|f_1\|_{k+2}^2, \|f_2\|_{k+2}^2 \right\}.$$

Proof of Lemma 2.3. We proceed by induction on k . The base case $k = 1$ is clear from Theorem 1.2. Now suppose the claim holds for $k = 1, 2, \dots, l$. Let $p(n)$ be a polynomial with degree $l + 1$. If $q_h(n) = p(n + h) - p(n)$, then $q_h(n)$ is a polynomial of degree less than or equal to l for all h , viewing n as the variable. By van der Corput's lemma and the Cauchy-Schwarz inequality, we know that

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_{l+1}[t]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|^2 \\ & \leq \frac{2}{H+1} + \frac{4}{H+1} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \sup_{q_h \in \mathbb{R}_l[t]} \left| \frac{1}{N} \sum_{n=1}^{N-h-1} (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(q_h(n)) \right| \\ & \leq \frac{2}{H+1} + 4 \left(\frac{1}{H+1} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \sup_{q_h \in \mathbb{R}_l[t]} \left| \frac{1}{N} \sum_{n=1}^{N-h-1} (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(q_h(n)) \right|^2 \right)^{1/2}. \end{aligned}$$

By integrating both sides (which is possible by Lemma 2.2) and applying Hölder's inequality, we have

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_{l+1}[t]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|^2 d\mu(x) \\ & \leq \frac{2}{H+1} + 4 \left(\frac{1}{H+1} \sum_{h=1}^H \int \limsup_{N \rightarrow \infty} \sup_{q_h \in \mathbb{R}_l[t]} \left| \frac{1}{N} \sum_{n=1}^{N-h-1} (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(q_h(n)) \right|^2 d\mu \right)^{1/2}. \end{aligned}$$

For any $1 \leq h \leq H$, the inductive hypothesis tells us that

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{q_h \in \mathbb{R}_l[t]} \left| \frac{1}{N} \sum_{n=1}^{N-h-1} (f_1 \cdot f_1 \circ T^{ah})(T^{an}x) (f_2 \cdot f_2 \circ T^{bh})(T^{bn}x) e(q_h(n)) \right|^2 d\mu \\ & \lesssim_{a,b,l} \min \left\{ \left\| f_1 \cdot f_1 \circ T^{ah} \right\|_{l+2}^2, \left\| f_2 \cdot f_2 \circ T^{bh} \right\|_{l+2}^2 \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_{l+1}[t]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|^2 d\mu(x) \\ & \lesssim_{a,b,l} \frac{1}{H} + \min \left\{ \left(\frac{1}{H} \sum_{h=1}^H \left\| f_1 \cdot f_1 \circ T^{ah} \right\|_{l+2}^2 \right)^{1/2}, \left(\frac{1}{H} \sum_{h=1}^H \left\| f_2 \cdot f_2 \circ T^{bh} \right\|_{l+2}^2 \right)^{1/2} \right\} \\ & \lesssim_{a,b,l} \frac{1}{H} + \min \left\{ \left(\frac{1}{H} \sum_{h=1}^H \left\| f_1 \cdot f_1 \circ T^{ah} \right\|_{l+2}^{2^{l+2}} \right)^{2^{-(l+2)}}, \left(\frac{1}{H} \sum_{h=1}^H \left\| f_2 \cdot f_2 \circ T^{bh} \right\|_{l+2}^{2^{l+2}} \right)^{2^{-(l+2)}} \right\}, \end{aligned}$$

and if we let $H \rightarrow \infty$, we obtain

$$\int \limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_{l+1}[t]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right|^2 d\mu(x) \lesssim_{a,b,l} \min \left\{ \left\| f_1 \right\|_{l+3}^2, \left\| f_2 \right\|_{l+3}^2 \right\}.$$

□

Proof of (1) of Theorem 1.3. By our assumption, either f_1 or f_2 belongs to \mathcal{Z}_{k+1}^\perp , which implies that either $\left\| f_1 \right\|_{k+2}$ or $\left\| f_2 \right\|_{k+2}$ equals 0, hence the right hand side of the inequality (7) equals 0. Thus, there exists a set of full measure X_{f_1, f_2} such that for any $x \in X_{f_1, f_2}$ and $p \in \mathbb{R}_k[\xi]$, we have

$$\limsup_{N \rightarrow \infty} \sup_{p \in \mathbb{R}_k[\xi]} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| = 0.$$

□

2.2. Proofs of (2) and (3) of Theorem 1.3. In this section, we first prove (2) of Theorem 1.3. Then we use this, together with (1), to prove (3) of the same theorem.

First, we prove the following approximation lemma; this allows us to reduce our proof to the case where f_1 and f_2 are both continuous functions on an ergodic nilsystem. The following inequality will be useful when dominating the averages in norm: Given a measure-preserving system (X, \mathcal{F}, μ, T) and $F \in L^\alpha(\mu)$

for $\alpha \in (1, \infty)$, we have

$$(8) \quad \left\| \sup_N \frac{1}{N} \sum_{n=1}^N F(T^n x) \right\|_\alpha \leq \frac{\alpha}{\alpha - 1} \|F\|_\alpha.$$

This inequality can be obtained by using the maximal ergodic theorem (see, for example, [2, Theorem 1.8] for a proof).

Lemma 2.4. *Let (X, \mathcal{F}, μ, T) be a measure-preserving system, and a and b be distinct integers. Let $f_1, f_2 \in L^\infty(\mu)$. Suppose there exist two sequences of functions $(f_1^i)_i$ and $(f_2^i)_i$ in $L^\infty(\mu)$ such that $\|f_1^i\|_{L^\infty(\mu)} < M$ for some constant $M > 0$ for any $i \in \mathbb{N}$, $f_j^i \rightarrow f_j$ in $L^2(\mu)$ -norm as $i \rightarrow \infty$ for each $j = 1, 2$, and for each i , there exists a set of full measure X_i such that for any $x \in X_i$ and any $p \in \mathbb{R}_k[\xi]$ for each $k \in \mathbb{N}$, the averages*

$$\frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) f_2^i(T^{bn}x) e(p(n))$$

converge. Then there exists a set of full measure $X_\infty \subset X$ such that for any $x \in X_\infty$ and any $p \in \mathbb{R}_k[\xi]$, the averages

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n))$$

converge for each $k \in \mathbb{N}$.

Proof. For each $j = 1, 2$, we can write $f_j = (f_j - f_j^i) + f_j^i$ for each i , so we can rewrite the averages as follows:

$$(9) \quad \begin{aligned} W_N(f_1, f_2, x, p) &= \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) \\ &= \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) + \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) (f_2 - f_2^i)(T^{bn}x) e(p(n)) \\ &\quad + \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) f_2^i(T^{bn}x) e(p(n)). \end{aligned}$$

Ultimately, we would like to show that there exists a set of full measure $X_\infty \subset X$ such that for any $x \in X_\infty$,

$$(10) \quad \mathcal{L}_R(W_N(f_1, f_2, x, p)) = \sup_{p \in \mathbb{R}_k[\xi]} \left(\limsup_{N \rightarrow \infty} \operatorname{Re}(W_N(f_1, f_2, x, p)) - \liminf_{N \rightarrow \infty} \operatorname{Re}(W_N(f_1, f_2, x, p)) \right) = 0,$$

and

$$(11) \quad \mathcal{L}_I(W_N(f_1, f_2, x, p)) = \sup_{p \in \mathbb{R}_k[\xi]} \left(\limsup_{N \rightarrow \infty} \operatorname{Im}(W_N(f_1, f_2, x, p)) - \liminf_{N \rightarrow \infty} \operatorname{Im}(W_N(f_1, f_2, x, p)) \right) = 0.$$

To show (10), we first note that the third term on the right hand side of (9) vanishes for μ -a.e. $x \in X$ for each i after applying \mathcal{L}_R since we know that the averages converge for all $x \in \bigcap_{i=1}^\infty X_i$, which is a set of full measure, and for any $p \in \mathbb{R}_k[\xi]$ and $i \in \mathbb{N}$. To show the remaining terms vanish, we apply Hölder's

inequality as well as the inequality (8). For instance, for the first term of (9), we have that

$$\begin{aligned} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| &\leq \frac{1}{N} \sum_{n=1}^N \left| (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) \right| \\ &\leq \|f_2\|_{L^\infty(\mu)} \frac{1}{N} \sum_{n=1}^N \left| f_1 - f_1^i \right| (T^{an}x). \end{aligned}$$

If we take supremum over N on both sides, we would have

$$\sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| \leq \|f_2\|_{L^\infty(\mu)} \left(\sup_{N \geq 1} \frac{1}{N} \sum_{n=1}^N \left| f_1 - f_1^i \right| (T^{an}x) \right),$$

so we integrate both sides (which is possible by Lemma 2.2) and apply Hölder's inequality for the right hand side to obtain

$$\int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| d\mu(x) \leq \|f_2\|_{L^\infty(\mu)} \left\| \sup_{N \geq 1} \frac{1}{N} \sum_{n=1}^N \left| f_1 - f_1^i \right| (T^{an}x) \right\|_{L^2(\mu)}.$$

We apply the inequality (8) for the case where $\alpha = 2$ to the $L^2(\mu)$ -norm on the right hand side to obtain

$$(12) \quad \int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| d\mu(x) \leq 2 \|f_2\|_{L^\infty(\mu)} \|f_1 - f_1^i\|_{L^2(\mu)}.$$

By the similar argument as in the first term of (9) (and recalling that $\|f_1^i\|_{L^\infty(\mu)} \leq M$ for all i), we can also obtain an estimate for the second term:

$$(13) \quad \int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) (f_2 - f_2^i)(T^{bn}x) e(p(n)) \right| d\mu(x) \leq 2M \|f_2 - f_2^i\|_{L^2(\mu)}.$$

We are now ready to verify (10). We note that

$$\begin{aligned} 0 &\leq \sup_{p \in \mathbb{R}_k[\zeta]} \left(\limsup_{N \rightarrow \infty} \operatorname{Re} (W_N(f_1, f_2, x, p)) - \liminf_{N \rightarrow \infty} \operatorname{Re} (W_N(f_1, f_2, x, p)) \right) \\ &\leq 2 \liminf_{i \rightarrow \infty} \left(\sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| \right. \\ &\quad \left. + \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) (f_2 - f_2^i)(T^{bn}x) e(p(n)) \right| \right). \end{aligned}$$

According to the inequalities (12) and (13), the integral of each average in the right-hand side of the inequality above is bounded by a constant multiple of either $\|f_1 - f_1^i\|_{L^2(\mu)}$ or $\|f_2 - f_2^i\|_{L^2(\mu)}$. These norms converge to 0 as $i \rightarrow \infty$. Using those inequalities together with Fatou's lemma, we obtain

$$\begin{aligned} (14) \quad &\int \sup_{p \in \mathbb{R}_k[\zeta]} \left(\limsup_{N \rightarrow \infty} \operatorname{Re} (W_N(f_1, f_2, x, p)) - \liminf_{N \rightarrow \infty} \operatorname{Re} (W_N(f_1, f_2, x, p)) \right) d\mu \\ &\leq 2 \liminf_{i \rightarrow \infty} \left(\int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N (f_1 - f_1^i)(T^{an}x) f_2(T^{bn}x) e(p(n)) \right| d\mu(x) \right. \\ &\quad \left. + \int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x) (f_2 - f_2^i)(T^{bn}x) e(p(n)) \right| d\mu(x) \right) \end{aligned}$$

$$+ \int \sup_{N \geq 1} \sup_{p \in \mathbb{R}_k[\zeta]} \left| \frac{1}{N} \sum_{n=1}^N f_1^i(T^{an}x)(f_2 - f_2^i)(T^{bn}x)e(p(n)) \right| d\mu(x) = 0.$$

Since inside the integral of (14) is nonnegative, (10) is established for μ -a.e. $x \in X$. Therefore, there exists a set of full measure $X_R \subset \bigcap_{i=1}^\infty X_i$ such that for any $x \in X_R$ and any polynomial $p \in \mathbb{R}_k[\zeta]$, the real part of the sequence $(W_N(f_1, f_2, x, p))_N$ converge for any $k \in \mathbb{N}$.

Similarly, we can show that (11) holds for μ -a.e. $x \in \bigcap_{i=1}^\infty X_i$, so there exists a set of full measure $X_I \subset \bigcap_{i=1}^\infty X_i$ such that for any $x \in X_I$ and any polynomial $p \in \mathbb{R}_k[\zeta]$, the imaginary part of the sequence $(W_N(f_1, f_2, x, p))_N$ converge for any $k \in \mathbb{N}$. So if we set $X_\infty = X_R \cap X_I$, we obtain the desired set of full measure. \square

To prove (2) of Theorem 1.3, we first prove this for the case where (X, \mathcal{F}, μ, T) is an ergodic nilsystem, and f_1 and f_2 are both continuous functions on X ; under these assumptions, the averages converge for all $x \in X$. The key ingredient of this proof is Leibman's pointwise convergence theorem of polynomial actions on a nilsystem [18], which is used to prove the following lemma.

Lemma 2.5. *Let (X, \mathcal{F}, μ, T) be a $(k+1)$ -step ergodic nilsystem. Suppose $f_1, f_2 \in \mathcal{C}(X)$, and $a, b \in \mathbb{Z}$ such that $a \neq b$. Then for any $x \in X$ and $p \in \mathbb{R}_k[\zeta]$, the averages*

$$(15) \quad \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x)f_2(T^{bn}x)e(p(n))$$

converge as $N \rightarrow \infty$.

Proof of Lemma 2.5. Let t be any real number. Suppose first that we fix a polynomial $q \in \mathbb{R}[\zeta]$. Suppose also that X is a $(k+1)$ -step nilsystem. Since we know that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a one-step nilmanifold, the system $(\mathbb{T}, \mathcal{B}, m, R_t)$ is a nilsystem, where \mathcal{B} is a Borel σ -algebra of \mathbb{T} , m is the usual Borel probability measure, and R_t is a rotation by t i.e. for any $\alpha \in (0, 1]$, we have $R_t(e(\alpha)) = e(\alpha + t)$. Thus, $(X^2 \times \mathbb{T}, \mathcal{F}^2 \otimes \mathcal{B}, \mu^2 \otimes m)$ is a $k+1$ -step nilmanifold. Suppose $F : X^2 \times \mathbb{T} \rightarrow \mathbb{C}$ for which

$$F(x_1, x_2, e(\alpha)) = f_1(x_1)f_2(x_2)e(\alpha).$$

Then F is continuous on $X^2 \times \mathbb{T}$. Hence, for any $\alpha \in [0, 1)$, we see that

$$\frac{1}{N} \sum_{n=1}^N F(T^{an}x_1, T^{bn}x_2, R_t^{q(n)}e(\alpha)) = \frac{e(\alpha)}{N} \sum_{n=1}^N f_1(T^{an}x_1)f_2(T^{bn}x_2)e(q(n)t).$$

Note that the left-hand side of the equation above converges by Leibman's convergence result for all $(x_1, x_2, e(\alpha)) \in X^2 \times \mathbb{T}$ [18, Theorems A, B]¹, so the averages in (15) converge for all $(x_1, x_2, e(\alpha)) \in X^2 \times \mathbb{T}$ for any $t \in \mathbb{R}$. In particular, it converges when $x_1 = x_2 = x$ and $\alpha = 0$, so we have shown that for any

¹In Leibman's paper, the polynomial sequences are defined for polynomials with integer coefficients. However, the cited theorems are proven for the case where one has a polynomial with real coefficients, as it is mentioned in [18, §3.13], provided that this action makes sense. Since the element of the group corresponding to the rotation in \mathbb{T} belongs to the identity component of the group, a real polynomial exponential makes sense.

$x \in X$, the averages

$$(16) \quad \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(tq(n))$$

converge as $N \rightarrow \infty$.

Now we fix $t \in (0, 1)$. Then for any k -th degree polynomial p , there exists another k -th degree polynomial q such that $p = tq$ (e.g. if $p(n) = \sum_{l=0}^k c_l n^l$, then we can set $q(n) = \sum_{l=0}^k c'_l n^l$, where $c'_l = c_l/t$). Thus, we have

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n)) = \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(tq(n)),$$

and we know that the averages in the right hand side is (16), so the averages on the left hand side converge for all $x \in X$ as $N \rightarrow \infty$. Hence, we have shown that the claim holds for the case where f_1 and f_2 are both continuous functions on an ergodic nilsystem. \square

Using the preceding lemma as well as Lemma 2.4, we prove (2) of Theorem 1.3.

Proof of (2) of Theorem 1.3. By the structure theorem [15, Theorem 10.1], the $k+1$ -th Host-Kra-Ziegler factor is the inverse limit of a sequence of $k+1$ -step ergodic nilsystems that are factor of (X, \mathcal{F}, μ, T) . This implies that if $f_1 \in \mathcal{Z}_{k+1}$, then for any $k+1$ -step ergodic nilsystem (N, \mathcal{N}, μ, T) that is a factor of (X, \mathcal{F}, μ, T) , we have $\|\mathbb{E}(f_1|\mathcal{N})\|_{L^\infty(\mu)} \leq \|f_1\|_{L^\infty(\mu)}$. Hence, by Lemma 2.4, it suffices to show that the statement of the theorem holds for the case where f_1 and f_2 are bounded and measurable with respect to a $k+1$ -step ergodic nilsystem (N, \mathcal{N}, μ, T) that is a factor of (X, \mathcal{F}, μ, T) . But since we know that if g_1 and g_2 are continuous functions on N , then the averages

$$\frac{1}{N} \sum_{n=1}^N g_1(T^{an}x) g_2(T^{bn}x) e(p(n))$$

converge for all $x \in N$ and $p \in \mathbb{R}_k[\xi]$ by Lemma 2.5. Furthermore, by density, there exist sequences of continuous functions $(\tilde{g}_1^i)_i$ and $(\tilde{g}_2^i)_i$ on N such that $\tilde{g}_j^i \rightarrow f_j$ in $L^2(\mu)$ as $i \rightarrow \infty$ for each $j = 1, 2$. We can construct another sequence of continuous functions $(g_1^i)_i$ such that

$$g_1^i(x) = \begin{cases} \min(\tilde{g}_1^i(x), \|f_1\|_{L^\infty(\mu)}) & \text{if } \tilde{g}_1^i(x) \geq 0, \\ \max(\tilde{g}_1^i(x), -\|f_1\|_{L^\infty(\mu)}) & \text{if } \tilde{g}_1^i(x) < 0, \end{cases}$$

so that $g_1^i \rightarrow f_1$ in $L^2(\mu)$ as $i \rightarrow \infty$, and $\|g_1^i\|_{L^\infty(\mu)} < \|f_1\|_{L^\infty(\mu)}$ for each $i \in \mathbb{N}$. Thus, we can apply Lemma 2.4 again (for the sequences $(g_1^i)_i$ and $(\tilde{g}_2^i)_i$) to show that the statement of the theorem holds for the case where f_1 and f_2 are bounded and measurable functions on N . \square

Now we are ready to prove (3) of Theorem 1.3 using (1) and (2).

Proof of (3) of Theorem 1.3. Since any continuous function ϕ on \mathbb{T} can be approximated by a linear combination of complex trigonometric functions, it suffices to prove this claim by showing that the averages

$$(17) \quad \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e(p(n))$$

converge off a single null-set independent of p . First we find a single set of full measure independent of $p \in \mathbb{R}_k[\xi]$ for which the averages converge. For each $j = 1, 2$, we write $f_j = \mathbb{E}(f_j | \mathcal{Z}_{k+1}) + f_j^\perp$, where $f_j^\perp \in \mathcal{Z}_{k+1}^\perp$. By (1) of Theorem 1.3, we merely need to show that the averages

$$(18) \quad \frac{1}{N} \sum_{n=1}^N \mathbb{E}(f_1 | \mathcal{Z}_{k+1})(T^{an}x) \mathbb{E}(f_2 | \mathcal{Z}_{k+1})(T^{bn}x) e(p(n))$$

converge on a set of full measure independent of $p \in \mathbb{R}_k[\xi]$. By (2) of Theorem 1.3, we know that there exists a set of full measure $X_{f_1, f_2, k}$ such that for any $x \in X_{f_1, f_2, k}$, the averages in (18) converges for all $p \in \mathbb{R}_k[\xi]$. Thus, if we set

$$X_{f_1, f_2} = \bigcap_{k=1}^{\infty} X_{f_1, f_2, k},$$

then X_{f_1, f_2} is a set of full measure independent for which the averages in (17) converge for all $x \in X_{f_1, f_2}$ and for all polynomials p with real coefficients. \square

3. COROLLARIES

One can directly prove the following weaker version of Theorem 1.3. We originally proved this result before we obtained the proof of Theorem 1.3, and used different approaches to prove a certain matter (e.g. Anzai's skew-product transformation on \mathbb{T} [1]). See [5] for more detail.

Corollary 3.1. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system, $a, b \in \mathbb{Z}$ such that $a \neq b$, and $f_1, f_2 \in L^2(X)$. Suppose $p(n)$ is a degree- k polynomial with real coefficients, where $k \geq 1$. Let*

$$W_N(f_1, f_2, x, p, t) = \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i p(n)t}.$$

(1) *If either f_1 or f_2 belongs to \mathcal{Z}_{k+1}^\perp , then there exists a set of full measure $X_{f_1, f_2, p}$ such that for all $x \in X_{f_1, f_2, p}$,*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, p, t)| = 0.$$

(2) *If $f_1, f_2 \in \mathcal{Z}_{k+1}$, then there exists a set of full measure $X_{f_1, f_2, p}$ such that for all $x \in X_{f_1, f_2, p}$, the averages $W_N(f_1, f_2, x, p, t)$ converge for all $t \in \mathbb{R}$.*

Proof. If p is a k -th degree polynomial with real coefficients, then so is tp , so both (1) and (2) hold if x belongs to the set of full measure obtained in Theorem 1.3. \square

The direct consequence of Corollary 3.1 is that the sequence $u_n = f_1(T^{an}x) f_2(T^{bn}x)$ is a universally good weight for the polynomial return time averages in norm.

Corollary 3.2. *Let (X, \mathcal{F}, μ, T) be an ergodic system, $a, b \in \mathbb{Z}$ such that $a \neq b$. Then there exists a set of full measure \tilde{X} such that for any $x \in \tilde{X}$, a polynomial $p \in \mathbb{Z}[\xi]$, and for any measure-preserving system (Y, \mathcal{G}, ν, S) and $g \in L^\infty(\nu)$, the averages*

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) g \circ S^{p(n)}$$

converge in $L^2(\nu)$.

Proof. This is a direct application of Theorem 1.3 and the spectral theorem. □

ACKNOWLEDGMENT

We thank Benjamin Weiss for suggesting to look at the extension of the double recurrence Wiener-Wintner result to nilsequences during the 2014 Ergodic Theory Workshop at UNC Chapel Hill. We also thank El Houcein El Abdalaoui for his interest in this problem. Lastly, we thank the anonymous referee for his/her suggestions and comments.

REFERENCES

- [1] H. Anzai. Ergodic skew product transformations on the torus. *Osaka Mathematical Journal*, 3(1):83–99, 1951.
- [2] I. Assani. *Wiener Wintner Ergodic Theorems*. World Science Pub Co Inc, May 2003.
- [3] I. Assani. Pointwise double recurrence and nilsequences. Preprint. Available from arXiv:1504.05732, 2015.
- [4] I. Assani, D. Duncan, and R. Moore. Pointwise characteristic factors for Wiener-Wintner double recurrence theorem. *Ergod. Th. and Dynam. Sys.*, 2015. Available on CJO 2015 doi:10.1017/etds.2014.99.
- [5] I. Assani and R. Moore. Extension of Wiener-Wintner double recurrence theorem to polynomials. Preliminary version, arXiv:1408.3064, 2014.
- [6] I. Assani and R. Moore. Extension of Wiener-Wintner double recurrence theorem to polynomials II. Preprint, arXiv:1409.0463, 2014.
- [7] I. Assani and R. Moore. A good universal weight for nonconventional ergodic averages in norm. To appear on *Ergod. Th. and Dynam. Sys.*, available on arXiv:1503.08863, 2015.
- [8] I. Assani and R. Moore. A good universal weight for multiple recurrence averages with commuting transformations in norm. Preprint, available on arXiv:1506.05370, June 2015.
- [9] V. Bergelson, B. Host, and B. Kra with appendix by I Ruzsa. Multiple recurrence and nilsequences. *Invent. Math.*, 160:261–303, 2005.
- [10] J. Bourgain. Double recurrence and almost sure convergence. *J. reine angew. Math.*, 404:140–161, 1990.
- [11] T. Eisner and B. Krause. (Uniform) convergence of twisted ergodic averages. To appear on *Ergod. Th. and Dynam. Sys.* Preprint available from arXiv:1407.4736, 2014.
- [12] T. Eisner and P. Zorin-Kranich. Uniformity in the Wiener-Wintner theorem for nilsequences. *Discrete and Continuous Dynamical Systems*, 33(8):3497–3516, 2013.
- [13] N. Frantzikinakis. Uniformity in the polynomial Wiener-Wintner theorem. *Ergod. Th. and Dynam. Sys.*, 26(4):1061–1071, 2006.
- [14] W. T. Gowers. A new proof of Szemerédi’s theorem. *Geom. Funct. Anal.*, 11:465–588, 2001.
- [15] B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. of Math.*, 161:387–488, 2005.
- [16] B. Host and B. Kra. Uniformity seminorms on ℓ^∞ and applications. *J. Anal. Math.*, 108:219–276, 2009.
- [17] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. John Wiley and Sons, 1974.

- [18] A. Leibman. Pointwise convergence of ergodic averages for polynomial sequence of translations on a nilmanifold. *Ergod. Th. and Dynam. Sys.*, 25:201–213, 2005.
- [19] E. Lesigne. Un théorème de disjonction de systèmes dynamiques et une généralisation du théorème ergodique de Wiener-Wintner. *Ergod. Th. and Dynam. Sys.*, 10:513–521, 1990.
- [20] E. Lesigne. Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polynômes. *Ergod. Th. and Dynam. Sys.*, 13:767–784, 1993.
- [21] T. Ziegler. Universal characteristic factors and Furstenberg averages. *J. Amer. Math. Soc.*, 20(1):53–97, 2006.
- [22] P. Zorin-Kranich. A nilsequence Wiener-Wintner theorem for bilinear ergodic averages. Preprint. Available from arXiv:1504.04647, 2015.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CHAPEL HILL, NC 27599

E-mail address: `assani@math.unc.edu`

URL: `http://www.unc.edu/math/Faculty/assani/`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CHAPEL HILL, NC 27599

E-mail address: `ryom@live.unc.edu`

URL: `http://ryom.web.unc.edu`